# On The Origin of Neutrino Mass and Mixing in the Standard Model

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One can describe cosmological relic neutrinos by adding Lagrange multipliers to the Standard Model Lagrangian for them. The two possible Lagrange multipliers are a chemical potential, which fixes the mean neutrino/anti-neutrino asymmetry, and a Majorana mass, which fixes the mean spin-entropy. Because these neutrinos originated from a thermal bath, their entropy should be maximal, implying that each state in the background is a symmetric superposition of a neutrino and anti-neutrino. Therefore the Standard Model must be augmented by a flavor-diagonal Majorana neutrino mass matrix. This impacts the propagator via tadpole diagrams due to self-interactions. In the low-energy limit, neutrino self-interactions are entirely off-diagonal because same-flavor four-fermion operators vanish by Pauli exclusion. These interactions must be diagonalized when propagating through a bath of neutrinos, using the $U(3)$ global flavor symmetry. $U(3)$ gets broken broken down to $SO(3)$ by Majorana masses, and down to $A_4$ if the three masses are different. Thus our universe today contains tri-bimaximal mixing and Majorana neutrinos. Neutrino mixing is due to the mismatch between the flavor-diagonal Majorana mass matrix arising at finite density and the self-interaction diagonal finite density propagator. The mass hierarchy is inverted and Majorana phases are absent. Lepton number is conserved and the neutrino-less double beta decay experiment absorbs a pair of neutrinos from the relic background and will prove their Majorana nature.

## INTRODUCTION

Neutrinos have been observed to change flavor. Originally known as the "solar neutrino problem" due to the apparent deficit of solar neutrinos, today this is well-understood as the mixing of electron-type neutrinos emitted in nuclear reactions into the $\mu$ and $\tau$ types. The most economical way to describe this mixing is via an off-diagonal mass matrix. One can implement this with either Majorana or Dirac neutrino masses. We currently have no experimental evidence that neutrino masses are either Dirac or Majorana, though there are several running and planned experiments to test this using neutrino-less double beta decay, which can only occur with a Majorana mass.[1] The current experimental situation is well-explained by a three-neutrino mixing hypothesis.[2]

The standard cosmological model predicts the existence of a sea of relic neutrinos if the temperature was ever above a few MeV. The leading order dynamics of this background that is relevant for neutrino experiments is its Pauli blocking. This can be included by adding Lagrange multipliers to the Lagrangian that fix the number density and entropy.

Whenever particles propagate in a medium, an index of refraction and rescaling of the kinetic term are induced, defined by the tadpole diagram corrections to the self-energy. This is a non-perturbative effect that must be re-summed using the Dyson series into the two-point function.

## MEAN FIELD NEUTRINOS

Our universe contains neutrinos, therefore when doing "vacuum" physics we should fix the parameters of the background relics with which we are not scattering. This is the approach taken whenever one adds a chemical potential directly to the Lagrangian. One is averaging over the degrees of freedom represented by the chemical potential, treating them instead as a "mean" field. One can also envision this as constraining the first moment of the thermal distribution function.

A finite density system in an arbitrary configuration can be represented by a density matrix

$$\rho_{ij}(p) = |\psi_i(p)\rangle\langle\psi_j(p)|. \tag{1}$$

The indices $i, j$ represent the distinct quantum numbers possible for a single momentum mode $p$, such as helicity and flavor. The average value of this density matrix is

$$\langle\rho_{ij}\rangle = \int \frac{d^3p}{(2\pi)^3\sqrt{2E}}\rho_{ij}(p), \tag{2}$$

In the mean field approach one assumes that the background state $\rho_{ij}$ is independent of time. By then adding Lagrange multipliers for the components of $\rho_{ij}(p)$ (written in terms of the field operator), one ensures that the result Eq.2 is reflected in the dynamics. This assumes that Eq.2 will be unchanged by the dynamics we are interested in computing.

Let us consider a single massless Weyl fermion. Its $2 \times 2$ spin-density matrix operator $\hat{\rho}$ can be related to the field operators using

$$v = (a_+,\ a_-); \qquad \hat{\rho} = v^\dagger|0\rangle\langle 0|v \tag{3}$$

where $a_\pm$ are the annihilation operators for the two helicities, and the momentum dependence has been suppressed for notational simplicity. The field operators for a Weyl fermion are

$$\chi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E}}\left[a_-u_-e^{-ipx} + a_+^\dagger v_+ e^{ipx}\right] \tag{4}$$

with the commuting state vectors in the helicity basis for neutrinos and anti-neutrinos respectively

$$u_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad v_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{5}$$

In terms of the density operator Eq.3, we can find the relationship between the density matrix and Lagrangian operators by evaluating $\mathrm{Tr}[\hat{\rho}\,\sigma^j]$:

$$\begin{aligned}
\mathrm{Tr}[\hat{\rho}\,\sigma^0] &= 1 \\
\mathrm{Tr}[\hat{\rho}\,\sigma^1] &= \frac{1}{2}\left[\chi^\dagger i\sigma^2\chi^* - \chi^T i\sigma^2\chi\right] = a_-^\dagger a_+ + a_+^\dagger a_- \\
\mathrm{Tr}[\hat{\rho}\,\sigma^2] &= \frac{1}{2}\left[\chi^\dagger \sigma^2\chi^* + \chi^T \sigma^2\chi\right] = i(a_+^\dagger a_- - a_-^\dagger a_+) \\
\mathrm{Tr}[\hat{\rho}\,\sigma^3] &= \chi^\dagger\sigma^0\chi = a_+^\dagger a_+ - a_-^\dagger a_-.
\end{aligned} \tag{6}$$

The first of these specifies the normalization of the density matrix. The second and third combinations are both Majorana masses, and are related by a lepton number phase transformation. Finally the fourth is the number operator.

In the mean field approximation described by the momentum-averaged density matrix, Eq.2, one can now add $\mu_j \mathrm{Tr}[\hat{\rho}\,\sigma_j]$ to the Lagrangian with constant coefficients. This is a Lagrange multiplier, and when finding the equations of motion for the fermion $\chi$, will ensure that these average values specified by the density matrix are preserved. Note that $\mu_i$ does not transform as a Lorentz vector.

Let us examine further the nature of these operators. With two neutrinos in a volume $V$, the averaged density matrix is

$$\langle\rho\rangle_2 = \begin{pmatrix} R_2^2 & R_2\sqrt{1-R_2^2}e^{i\varphi_2} \\ R_2\sqrt{1-R_2^2}e^{-i\varphi_2} & 1-R_2^2 \end{pmatrix}\frac{2}{V}. \tag{7}$$

The appropriate traces analogous to Eq.6 are

$$\mathrm{Tr}[\langle\rho\rangle_2\sigma^1] = 2R_2\sqrt{1-R_2^2}\cos(\varphi_2)\frac{2}{V} \tag{8}$$

$$\mathrm{Tr}[\langle\rho\rangle_2\sigma^2] = 2R_2\sqrt{1-R_2^2}\sin(\varphi_2)\frac{2}{V} \tag{9}$$

$$\mathrm{Tr}[\langle\rho\rangle_2\sigma^3] = (2R_2^2-1)\frac{2}{V}, \tag{10}$$

where we've taken a unit volume $V$. For a pure state of two neutrinos, $R_2 = 0$, the Majorana mass term $\mathrm{Tr}[\langle\rho\rangle_2\sigma^1] = 0$, and the chemical potential term $\mathrm{Tr}[\langle\rho\rangle_2\sigma^3] = -2/V$ is negative. For a pure state of anti-neutrinos, $R_2 = 1$ and the chemical potential term changes sign, while the Majorana mass term remains zero. For a mixed state with an equal probability for each to be a neutrino or anti-neutrino, $R_2 = 1/\sqrt{2}$, the Majorana mass terms are maximal, and the chemical potential term $\mathrm{Tr}[\langle\rho\rangle_2\sigma^3]$ is zero.

The chemical potential $\mu_3$ is expected to be related to the baryon asymmetry of the universe, and very small

$$\eta_\nu = \frac{n_\nu - n_{\overline{\nu}}}{n_\gamma} \sim 6\times10^{-10}.$$

Therefore we will neglect it and assume $\mu_3 = 0$ so that the number of neutrinos and anti-neutrinos are the same.

The state with both $\mu_1$ and $\mu_2$ zero is a superposition of a pure state of neutrinos with a pure state of anti-neutrinos. The averaged density matrix is proportional to the identity. This configuration has zero spin-entropy

$$S_\pm = -k\mathrm{Tr}[\langle\rho\rangle\ln\langle\rho\rangle]. \tag{11}$$

Since the early-universe plasma from which these neutrinos originated should be extremely close to a blackbody spectrum, as is observed for the Cosmic Microwave Background, the entropy for this system should instead be maximal. For the spin-entropy this corresponds to maximizing the magnitude of the off-diagonal components, e.g. $R = 1/\sqrt{2}$, and is a state that is in an equal superposition of neutrino and anti-neutrino.

If we examine the thermal Fermi-Dirac distribution $f(E) = (e^{E/kT}+1)^{-1}$ for a system with zero chemical potential, we see that $f(E) \le 1/2$ for all energies. In other words, each momentum mode there is either a neutrino or an anti-neutrino, but not both (which would correspond to $f(E) = 1$ for that energy). If both were present for a single momentum mode, Fermi statistics indicates that the two density matrices are orthogonal $\rho_1\cdot\rho_2 = 0$, which implies that their sum is the identity. This configuration has zero spin-entropy also.

Each pair in the symmetric bath therefore contributes $S_\pm = k\,2\ln 2$ to the spin-entropy. Expanding in Eq.11, $\langle\rho\rangle = 1+\epsilon$, where $\epsilon$ is off diagonal, we can see that only the magnitude of the off diagonal components of $\langle\rho\rangle$ contribute to the spin-entropy. If the phase were random, as in the Random Phase Approximation (RPA), these off diagonal elements would cancel in the sum, Eq.2 leading again to zero spin-entropy. This is clearly incorrect. Instead the condition of maximal spin-entropy implies that all states have a fixed, common phase.

The two masses $\mu_1$ and $\mu_2$ are related by the lepton number phase symmetry $\chi \to e^{i\eta}\chi$. The Standard Model conserves lepton number for each species individually, so we can use this freedom to rotate away one of $\mu_1$ or $\mu_2$. In other words, the Majorana phases $\alpha = \arctan(\mu_1/\mu_2)$ if present, can be set to zero and are unphysical. Only $\mu_1^2 + \mu_2^2$ is physical.

Thus given the expectation that the relic neutrino background has small chemical potential and maximal entropy, we should add to the Standard Model a Majorana mass (which we can choose to be $\mu_1$ or $\mu_2$) for each flavor, as well as a small chemical potential. We do not add flavor off-diagonal terms because the creation and annihilation operators in Eq.6 are related by the Lorentz representation of the Weyl field. In other words, a Lorentz transformation does not perform flavor rotations.

Thus, the Majorana mass should not be considered a "fundamental" Lagrangian parameter. Instead it is an

environmental parameter; a Lagrange multiplier analogous to the chemical potential for a system with spin-entropy. Because the spin-entropy is maximal and simply related to the number density, the Majorana mass is also in effect fixing the total number of neutrinos. Quantitatively then, it is simply the Fermi momentum

$$\mu_1 = p_F = (3\pi^2 n)^{1/3} \tag{12}$$

where $n$ is the number density.

## SELF-INTERACTIONS

Neutrino self-interactions occur at tree level by $Z$ boson exchange. The energies of experiments which have demonstrated neutrino mixing are far below the energy where the process $\nu\overline{\nu} \to \nu\overline{\nu}$ becomes large because the $Z$ boson is on shell. Therefore we may integrate the $Z$ out using its leading order equation of motion at zero momentum, which generates four-point neutrino contact operators.

For neutrino energies $E \ll M_Z^2/T_\nu$, only these interaction terms survive

$$\begin{aligned} J_i^\lambda (J_i)_\lambda &= (\chi_e^\dagger \overline{\sigma}^\lambda \chi_e + \chi_\mu^\dagger \overline{\sigma}^\lambda \chi_\mu + \chi_\tau^\dagger \overline{\sigma}^\lambda \chi_\tau)^2 \\ &= J_{e\mu}^\lambda (J_{e\mu})_\lambda + J_{\mu\tau}^\lambda (J_{\mu\tau})_\lambda + J_{\tau e}^\lambda (J_{\tau e})_\lambda \end{aligned} \tag{13}$$

in terms of the currents $J_i^\lambda = \epsilon^{ijk} \chi_j^\dagger \overline{\sigma}^\lambda \chi_k$. The same-flavor four-fermion interactions vanished in this equation because

$$\chi^\dagger \overline{\sigma}^\lambda \chi \chi^\dagger \overline{\sigma}_\lambda \chi = 0 \tag{14}$$

which is another way of saying that same-flavor interactions only occur in the $p$-wave due to Pauli blocking, and vanish in the low energy limit below the $Z$ mass.

The interactions of Eq.13 clearly do not correspond to the flavor-diagonal kinetic terms. Therefore we should do a basis rotation to make them flavor diagonal before computing the Green's functions, using the Standard Model's vacuum global symmetry $U(3)$ on the three families of leptons. Adding Majorana masses that are the same for all families breaks all the real generators of this symmetry. The remaining pure imaginary generators form a representation of $SO(3)$.

This residual $SO(3)$ of the Standard Model with three equal Majorana masses and its Noether current are

$$\chi_i \to e^{\alpha_a T_{ij}^a} \chi_j; \qquad J_i^\lambda = \epsilon_{ijk} \chi_j^\dagger \overline{\sigma}^\lambda \chi_k, \tag{15}$$

where the Latin indices $\{i, j\}$ run over the three families and the three generators $T^a$ are

$$T^{e\mu} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; T^{\mu\tau} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \; T^{\tau e} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

with $a = \{e\mu, \mu\tau, \tau e\}$, and we can see that the currents in Eq.15 are off-diagonal relative to the flavor basis,

$$\begin{aligned} J_{e\mu}^\lambda &= \chi_e^\dagger \overline{\sigma}^\lambda \chi_\mu - \chi_\mu^\dagger \overline{\sigma}^\lambda \chi_e \\ J_{\mu\tau}^\lambda &= \chi_\mu^\dagger \overline{\sigma}^\lambda \chi_\tau - \chi_\tau^\dagger \overline{\sigma}^\lambda \chi_\mu \\ J_{\tau e}^\lambda &= \chi_\tau^\dagger \overline{\sigma}^\lambda \chi_e - \chi_e^\dagger \overline{\sigma}^\lambda \chi_\tau . \end{aligned} \tag{16}$$

Using our vacuum flavor $SO(3)$ freedom, we can find a basis in which the interacting currents correspond to the kinetic term fields (which remain diagonal under a $SO(3)$ flavor rotation). Because the couplings are all the same there is nothing to differentiate the flavors in the vacuum, so we know that the diagonal interaction will take the form of a sum of squares of a current, and that current must be the same for all three fields. The rotation that accomplishes this is the "tri-bimaximal mixing" matrix, and has the form

$$U = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}. \tag{17}$$

The three identical currents after the rotation are

$$J_{\rm int}^\lambda = \frac{1}{\sqrt{3}} \left( J_{e\mu}^\lambda + J_{\mu\tau}^\lambda + J_{\tau e}^\lambda \right) \tag{18}$$

leading to an effective interaction

$$\begin{aligned} \frac{g_Z^2}{2M_Z^2} J_{\rm int}^\lambda J_\lambda^{\rm int} &= \frac{g_Z^2}{6M_Z^2} \sum_{i=1}^3 \chi_i^\dagger \overline{\sigma}^\lambda \chi_i \chi_i^\dagger \overline{\sigma}_\lambda \chi_i . \\ &= \frac{g_Z^2}{6M_Z^2} \sum_{i=1}^3 \psi_i^T \gamma^\lambda P_L \psi_i \psi^T \gamma_\lambda P_L \psi_i = 0. \end{aligned} \tag{19}$$

where we have written the expression using real Majorana spinors in the last line. This is the vacuum self-interaction basis of neutrinos in the Standard Model. However one can see by Eq.14 that because different flavor fermions also normally taken to anti-commute, this rotated four-Fermi operator vanishes too! Therefore, at $\mathcal{O}(G_F)$ neutrinos have no self-interactions.

Eq.19 has a discrete $A_4$ symmetry, in which the legs of the four-Fermi operator are exchanged (this is easier to see using real Majorana spinors). The discrete symmetry of the interchange of four objects generates the "tri-bimaximal" mixing matrix, Eq.17.

The interaction term of Eq.13 and Eq.19 are identical. Nothing in the vacuum Lagrangian forces us to choose this, or any other basis. It is the basis defined by $U$ that the neutrino self-interactions are diagonal in the low-energy limit, and the neutrino propagator is most transparently defined when propagating through a neutrino medium. One can think of neutrinos as sharing one propagator, even though they interact with charged currents as three separate fields.

Following the arguments of the previous section, we must now add a flavor-diagonal Majorana mass matrix to the Lagrangian. We chose the flavor basis because one should first write the vacuum Lagrangian in a basis with diagonal 2-point correlation functions for all fields (especially the charged leptons) and then add Lagrange multipliers corresponding to the degrees of freedom in that vacuum Lagrangian. Writing masses diagonally in any other basis is equivalent however.

This mass matrix does not have the $SO(3)$ symmetry of the vacuum Lagrangian, and breaks the $SO(3)$ down to the discrete $A_4$ in Eq.19. If two of the masses were equal there would be a remaining $SO(2)$ symmetry, but we already know experimentally this is not the case.

## NEUTRINO 2-POINT GREEN'S FUNCTION

The purpose of the analysis in the two preceding sections is to enable us to compute the full, in-medium two-point correlation function. This means computing the one-particle irreducible (1PI) tadpole interaction with the background. The effect of these tadpoles is inherently non-perturbative, and cannot be neglected due to a small coupling with the medium.

The propagation of a single fermion is governed by the equation of motion

$$[\not{k} - \Sigma(k)]\,\psi = 0. \tag{20}$$

The self-energy $\Sigma(k)$ has the general form

$$\Sigma(k) = m - a\not{k} + b\not{u} + c\,[\not{k}, \not{u}] \tag{21}$$

where $u^\alpha$ is the frame in which the thermal distribution function is defined.[3] Terms proportional to $u^\alpha$ can only show up at $\mathcal{O}(G_F^2)$ in the low energy theory, because the Majorana mass is Lorentz invariant. It is an expectation value for the energy $\langle E\rangle \neq 0$ that defines the rest frame of the medium $u^\alpha$. Here $a$ contributes to a rescaling of the kinetic term, and $b$ and $c$ contribute to the index of refraction.

The really dangerous term that necessitates the diagonalization of Eq.17 is $a$. If we had done these computations in a basis with an off-diagonal mass matrix (whatever its origin), and off-diagonal self-interactions we would have also generated off-diagonal kinetic terms, via the $a$ and $c$ of Eq.21 that must be diagonalized in order to have a sensible set of correlation functions.

Using the self-interaction basis in Eq.17, the neutral current interactions are entirely removed at $\mathcal{O}(G_F)$. The finite density contributions to Eq.21 therefore arise at $\mathcal{O}(G_F^2)$ due to charged $W^\pm$ bosons and $p$-wave $Z$ bosons. We have kept the kinetic term diagonal via this rotation at the expense of generating non-diagonal matrices for neutrino mass, neutrino chemical potential, and charged lepton mass. All three of these matrices break the $A_4$ symmetry that we found among the neutral interactions at $\mathcal{O}(G_F)$. Therefore, all three of these generate corrections to $U$ at $\mathcal{O}(G_F^2)$. The largest of these arises from the $\tau$ lepton, which generates corrections

$$\delta U \sim \mathcal{O}\left(\frac{m_\tau^2 G_F}{16\pi^2}\right) \sim 10^{-7} \tag{22}$$

The contributions at $\mathcal{O}(G_F)$ found in Refs.[4, 5] vanish by Eq.14 in the case of same-flavor interactions and are only present when considering a fictitious experiment of a neutrino propagating through a bath of neutrinos of a different flavor. Radiative corrections to these terms are also computed in Ref.[6].

## DISCUSSION

The presence of the relic neutrino background in the universe necessitates diagonalizing neutrino self-interactions in order to compute the neutrino propagator. The dominant observable consequence of this is an apparent flavor mixing due to the mismatch between the propagator's self-interaction basis and the flavor basis in which neutrinos are created today by $W^\pm$ bosons, and their spin-entropy defined.

The mixing matrix is given uniquely by the requirement that the interactions are diagonal, and is of "tri-bimaximal" form, previously guessed but not explained by other authors.[7, 8] Therefore

$$\sin^2\theta_{12} = 1/3 \quad \sin^2\theta_{23} = 1/2 \quad \sin^2\theta_{13} = 0 \tag{23}$$

in the Standard Model, in the absence of new physics, if there exists a relic neutrino background containing all three flavors. The leading corrections to this are $\mathcal{O}(m_\tau^2 G_F/16\pi^2) \sim 10^{-7}$.

In the mixing matrix $\sin^2\theta_{13} = 0$ also implies the absence of CP violation, which can be understood because the relic neutrino background with no chemical potential is itself C and CP symmetric. A tiny $\sin^2\theta_{13}$ will be generated by loops of charged leptons, and a further correction as well as an extremely tiny CP violating phase will be generated by a chemical potential $\mu_3$.

Because the $\tau^\pm$ freezes out of thermal equilibrium before the $\mu^\pm$ and $\pi^\pm$, the relic density of $\nu_\tau$ is smaller than that of $\nu_e$ and $\nu_\mu$ today. The $\pi^\pm$ and $\mu^\pm$ are very similar in mass, so they freeze out creating similar densities of $\nu_e$ and $\nu_\mu$. Thus due to cosmic expansion the density of $\nu_\tau$ has had more time to dilute than that of $\nu_e$ and $\nu_\mu$, so the hierarchy is inverted with $\delta m^2_{\rm sol} = m^2_{\nu_e} - m^2_{\nu_\mu}$ and $\delta m^2_{\rm atm} = m^2_{\nu_e} - m^2_{\nu_\tau}$.

We unambiguously predict that the Standard Model's neutrino mass is of Majorana type. Therefore neutrinoless double beta decay experiments will observe the neutrino mass, and can be understood as absorbing a pair of

neutrinos from the background. New physics which generates mixing is perturbing around tri-bimaximal mixing, rather than no mixing.

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[1] M. Doi, T. Kotani, and E. Takasugi. Double beta Decay and Majorana Neutrino. *Prog. Theor. Phys. Suppl.*, 83:1, 1985.

[2] Thomas Schwetz, M. A. Tortola, and Jose W. F. Valle. Three-flavour neutrino oscillation update. *New J. Phys.*, 10:113011, 2008.

[3] H. Arthur Weldon. Effective fermion masses of order gT in high-temperature gauge theories with exact chiral invariance. *Physical Review D*, 26(10):2789, November 1982. Copyright (C) 2009 The American Physical Society; Please report any problems to prola@aps.org.

[4] Dirk Notzold and Georg Raffelt. Neutrino dispersion at finite temperature and density. *Nuclear Physics B*, 307(4):924–936, October 1988.

[5] Juan Carlos D'Olivo, Jose F. Nieves, and Manuel Torres. Finite temperature corrections to the effective potential of neutrinos in a medium. *Phys. Rev.*, D46:1172–1179, 1992.

[6] Paul Langacker and Jiang Liu. Standard model contributions to the neutrino index of refraction in the early universe. *hep-ph/9206209*, June 1992. Phys.Rev. D46 (1992) 4140-4160.

[7] P. F. Harrison, D. H. Perkins, and W. G. Scott. A redetermination of the neutrino mass-squared difference in tri-maximal mixing with terrestrial matter effects. *Physics Letters B*, 458(1):79 – 92, 1999.

[8] P. F. Harrison, D. H. Perkins, and W. G. Scott. Tribimaximal mixing and the neutrino oscillation data. *Physics Letters B*, 530(1-4):167 – 173, 2002.